2.12 Which of the following sets are convex?

(a) A *slab*, *i.e.*, a set of the form $\{x \in \mathbf{R}^n \mid \alpha \leq a^T x \leq \beta\}$.

(b) A *rectangle*, *i.e.*, a set of the form $\{x \in \mathbf{R}^n \mid \alpha_i \leq x_i \leq \beta_i,\ i = 1, \ldots, n\}$. A rectangle is sometimes called a *hyperrectangle* when $n > 2$.

(c) A *wedge*, *i.e.*, $\{x \in \mathbf{R}^n \mid a_1^T x \leq b_1,\ a_2^T x \leq b_2\}$.

(d) The set of points closer to a given point than a given set, *i.e.*,

$$\{x \mid \|x - x_0\|_2 \leq \|x - y\|_2 \text{ for all } y \in S\}$$

where $S \subseteq \mathbf{R}^n$.

(e) The set of points closer to one set than another, *i.e.*,

$$\{x \mid \mathbf{dist}(x, S) \leq \mathbf{dist}(x, T)\},$$

where $S, T \subseteq \mathbf{R}^n$, and

$$\mathbf{dist}(x, S) = \inf\{\|x - z\|_2 \mid z \in S\}.$$

(f) [HUL93, volume 1, page 93] The set $\{x \mid x + S_2 \subseteq S_1\}$, where $S_1, S_2 \subseteq \mathbf{R}^n$ with S_1 convex.

(g) The set of points whose distance to a does not exceed a fixed fraction θ of the distance to b, *i.e.*, the set $\{x \mid \|x - a\|_2 \leq \theta \|x - b\|_2\}$. You can assume $a \neq b$ and $0 \leq \theta \leq 1$.

3.21 *Pointwise maximum and supremum.* Show that the following functions $f : \mathbf{R}^n \to \mathbf{R}$ are convex.

(a) $f(x) = \max_{i=1,\ldots,k} \|A^{(i)}x - b^{(i)}\|$, where $A^{(i)} \in \mathbf{R}^{m \times n}$, $b^{(i)} \in \mathbf{R}^m$ and $\|\cdot\|$ is a norm on $\mathbf{R}^m$.

(b) $f(x) = \sum_{i=1}^r |x|_{[i]}$ on $\mathbf{R}^n$, where $|x|$ denotes the vector with $|x|_i = |x_i|$ (*i.e.*, $|x|$ is the absolute value of x, componentwise), and $|x|_{[i]}$ is the ith largest component of $|x|$. In other words, $|x|_{[1]}, |x|_{[2]}, \ldots, |x|_{[n]}$ are the absolute values of the components of x, sorted in nonincreasing order.

3.32 *Products and ratios of convex functions.* In general the product or ratio of two convex functions is not convex. However, there are some results that apply to functions on $\mathbf{R}$. Prove the following.

(a) If f and g are convex, both nondecreasing (or nonincreasing), and positive functions on an interval, then fg is convex.

(b) If f, g are concave, positive, with one nondecreasing and the other nonincreasing, then fg is concave.

(c) If f is convex, nondecreasing, and positive, and g is concave, nonincreasing, and positive, then f/g is convex.

3.36 Derive the conjugates of the following functions.

(a) *Max function.* $f(x) = \max_{i=1,\ldots,n} x_i$ on $\mathbf{R}^n$.

(b) *Sum of largest elements.* $f(x) = \sum_{i=1}^r x_{[i]}$ on $\mathbf{R}^n$.

(c) *Piecewise-linear function on* $\mathbf{R}$. $f(x) = \max_{i=1,\ldots,m}(a_i x + b_i)$ on $\mathbf{R}$. You can assume that the a_i are sorted in increasing order, *i.e.*, $a_1 \leq \cdots \leq a_m$, and that none of the functions $a_i x + b_i$ is redundant, *i.e.*, for each k there is at least one x with $f(x) = a_k x + b_k$.

(d) *Power function.* $f(x) = x^p$ on $\mathbf{R}_{++}$, where $p > 1$. Repeat for $p < 0$.

(e) *Negative geometric mean.* $f(x) = -(\prod x_i)^{1/n}$ on $\mathbf{R}^n_{++}$.

(f) *Negative generalized logarithm for second-order cone.* $f(x, t) = -\log(t^2 - x^T x)$ on $\{(x, t) \in \mathbf{R}^n \times \mathbf{R} \mid \|x\|_2 < t\}$.