Exercices

2.12 (a) $E = \{x \in \mathbb{R}^n \mid \alpha \leq a^T x \leq \beta\}$ $\quad \alpha, \beta \in \mathbb{R}$, $a \in \mathbb{R}^n$

Let $x, y \in E$ and $0 \leq \theta \leq 1$

$$a^T(\theta x + (1-\theta)y) = \theta a^T x + (1-\theta) a^T y \quad \text{and } x, y \in E$$

$$\Rightarrow \alpha = \theta\alpha + (1-\theta)\alpha \leq a^T(\theta x + (1-\theta)y) \leq \theta\beta + (1-\theta)\beta = \beta$$

$$\Rightarrow \theta x + (1-\theta)y \in E$$

E is convex

(b) $E = \{x \in \mathbb{R}^n \mid \alpha_i \leq x_i \leq \beta_i,\ i = 1, \dots, n\}$ $\quad \alpha_i, \beta_i \in \mathbb{R}$

Let $x, y \in E$ and $0 \leq \theta \leq 1$. Let us note $z = \theta x + (1-\theta)y$

$z_i = \theta x_i + (1-\theta) y_i$ for $i = 1, \dots, n$

and $x, y \in E \Rightarrow \alpha \leq z_i \leq \beta$

$\Rightarrow z \in E$

E is convex

(c) $E = \{x \in \mathbb{R}^n \mid a_1^T x \leq b_1,\ a_2^T x \leq b_2\}$

Let $x, y \in E$ and $0 \leq \theta \leq 1$. Let us note $z = \theta x + (1-\theta)y$

$a_1^T z = \theta a_1^T x + (1-\theta) a_1^T y \leq b_1$, idem for $a_2^T z$

$\Rightarrow z \in E$

E is convex

(d) $E = \{x \mid \|x - x_0\|_2 \leq \|x - y\|_2 \quad \forall y \in S\}$ with $S \subseteq \mathbb{R}^n$

Let $y \in S$, and $E_y = \{x \mid \|x - x_0\|_2 \leq \|x - y\|_2\}$ $\quad (y \neq x_0)$

$$\|x - x_0\|_2 \leq \|x - y\|_2 \iff (x - x_0)^T(x - x_0) \leq (x - y)^T(x - y)$$

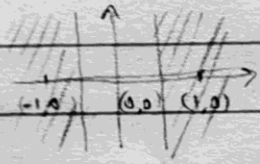
$$\iff 2(y - x_0)^T x \leq \|y\|_2^2 - \|x_0\|_2^2$$

Hence E_y is a halfspace which is convex.
If $y = x_0$, $E_y = \mathbb{R}^n$ which is convex
Therefore $E = \bigcap_{y \in S} E_y$ is an intersection of convex spaces.

<u>E is convex</u>

(e) $E = \{x \mid \mathrm{dist}(x,S) \leq \mathrm{dist}(x,T)\}$ where $S,T \subseteq \mathbb{R}^n$
and $\mathrm{dist}(x,S) = \inf\{\|x-z\|_2 \mid z \in S\}$

Let $n=2$, $S = \{(-1,0),(1,0)\}$ and $T = \{(0,0)\}$



$(-1,0)$ and $(0,1)$ belong to E.
but $(0,0) = \frac{1}{2}(-1,0) + (1-\frac{1}{2})(1,0)$ does not

<u>E is not always convex</u>
(we could have taken $n \in \mathbb{N}^*$ by using $S = \{(-1,0,\dots,0),(1,0,\dots,0)\}$)

(f) $E = \{x \mid x + S_2 \subseteq S_1\}$ $S_1, S_2 \subseteq \mathbb{R}^n$ with S_1 convex
Let $y \in S_2$ and $E_y = \{x \mid \underbrace{x+y}_{\text{affine of } x} \in S_1\}$

E_y is an affine transform of S_1 which is convex
$\Rightarrow E_y$ is convex
$E = \bigcap_{y \in S_2} E_y$ is an intersection of convex spaces

<u>E is convex</u>

(g) $E = \{x \mid \|x-a\|_2 \leq \theta \|x-b\|_2\}$ with $a \neq b$ and $0 \leq \theta \leq 1$
Let $x \in \mathbb{R}^n$.

One has $\|x-a\|_2 \leq \theta \|x-b\|_2 \iff (x-a)^T(x-a) \leq \theta^2 (x-b)^T(x-b)$

$$\iff \underbrace{(1-\theta^2)\, x^Tx + 2(b-a)^T x + (b^Tb - a^Ta)}_{f(x)} \leq 0$$

f is twice differentiable and one has $\nabla^2 f(x) = \underbrace{(1-\theta^2)}_{\geq 0} I_n$

hence $\nabla^2 f(x) \succeq 0$ for all $x \in \mathbb{R}^n$

therefore f is convex over $\mathbb{R}^n$

Given that $E = \{x \mid f(x) \leq 0\}$, E is a 0-sublevel set of f which is convex

E is convex

3.21 (a) $f(x) = \max_{i=1,\dots,k} \|A^{(i)}x - b^{(i)}\|$, $A^{(i)} \in \mathbb{R}^{m,n}$, $b^{(i)} \in \mathbb{R}^m$

$\forall i = 1, \dots, k$, $x \mapsto A^{(i)}x - b^{(i)}$ is affine

$y \mapsto \|y\|$ is convex (it is a norm)

Hence $\forall i = 1, \dots, k$ $f_i := x \mapsto \|A^{(i)}x - b^{(i)}\|$ is convex

Therefore $f = \max_{i=1,\dots,k} f_i$ is convex

(b) $f(x) = \sum_{i=1}^{r} |x|_{[i]}$

Let $i = 1, \dots, n$, $x \mapsto |x|_i$ is convex (since it is the composition of $\underbrace{x \mapsto |x|}_{\text{convex}}$ and $\underbrace{x \mapsto x_i}_{\text{affine}}$)

Hence $\forall 1 \leq i_1 < i_2 < \dots < i_r \leq n$ $x \mapsto \sum_{i=1}^{r} |x|_{i_r}$ is convex

But $\forall x \in \mathbb{R}^n$ $f(x) = \max_{\{i_1,\dots,i_r \mid 1 \leq i_1 < \dots < i_r \leq n\}} \sum_{i=1}^{r} |x|_{i_r}$

Therefore f is convex (pointwise maximum of $n!/r!(n-r)!$)

3.32 (a) f, g convex $\left\{\begin{array}{l}\text{or nondecreasing}\\ \text{nonincreasing}\end{array}\right.$, positive on an interval I, then fg convex

Let $0 \leq \theta \leq 1$ and $x, y \in I$

$f(\theta x + (1-\theta)y)\, g(\theta x + (1-\theta)y)$

$\leq (\theta f(x) + (1-\theta) f(y))(\theta g(x) + (1-\theta) g(y))$ since $f, g \geq 0$ all products are ≥ 0

$= \theta^2 f(x) g(x) + \theta(1-\theta) f(y) g(x) + \theta(1-\theta) f(x) g(y) + (1-\theta)^2 f(y) g(y)$

$-\theta f(x) g(x) + \theta f(x) g(x) \qquad + (1-\theta) f(y) g(y) - (1-\theta) f(y) g(y)$

$= \theta f(x) g(x) + (1-\theta) f(y) g(y)$

$+ \theta(\theta - 1)\left[f(x) g(x) - f(y) g(x) - f(x) g(y) + f(y) g(y) \right]$

$= \theta f(x) g(x) + (1-\theta) f(y) g(y)$

$+ \theta \underbrace{(\theta - 1)}_{\leq 0} \left[\underbrace{(f(x) - f(y))(g(x) - g(y))}_{\geq 0} \right]$ since f, g are both increasing or both decreasing

$\leq \theta f(x) g(x) + (1-\theta) f(y) g(y)$

Therefore $\forall x, y \in I$, $\forall\, 0 \leq \theta \leq 1$, $fg(\theta x + (1-\theta)y) \leq \theta fg(x) + (1-\theta) fg(y)$

Hence fg is convex

(b) By an analogous reasoning, one just needs to reverse the inequality of the first step

Then, $\theta \underbrace{(\theta - 1)}_{\leq 0} \left[\underbrace{(f(x) - f(y))(g(x) - g(y))}_{\leq 0} \right] \geq 0$ at the final step

Which proves that fg is concave

(c) If g is concave and positive, $1/g$ is convex, then because g is $\left\{\begin{array}{l}\text{nonincreasing}\\ \text{and positive}\end{array}\right.$, $1/g$ is $\left\{\begin{array}{l}\text{nondecreasing}\\ \text{and positive}\end{array}\right.$

Therefore, (a) applied to f and $1/g$ gives that f/g is convex

3.36. Conjugates of:

(a) $f(x) = \max_i x_i \quad x \in \mathbb{R}^n$

Let $y \in \mathbb{R}^n$

- if $\exists k \in \{1, \dots, n\}$ such that $y_k < 0$

  Let $x_t = (0, \dots, 0, -t, 0, \dots, 0)$ (k^{th} component)

  $$y^T x - f(x) = -t y_k \xrightarrow[t \to +\infty]{} +\infty \Rightarrow \underline{f^*(y) = +\infty}$$

- now, suppose $y \succeq 0$. One has $y^T x - f(x) = y_1 x_1 + \dots + y_n x_n - \max x_i$

  if $\sum_{i=1}^{n} y_i > 1$, then for $x_t = (t, \dots, t)$

  $$y^T x - f(x) = \sum y_i t - t = \underbrace{(\sum y_i - 1)}_{> 0} t \xrightarrow[t \to +\infty]{} +\infty$$

  then $\underline{f^*(y) = +\infty}$

- now, suppose $y \succeq 0$ and $\sum_{i=1}^{m} y_i < 1$.

  for $x_t = (-t, \dots, -t)$ $\quad y^T x - f(x) = -\underbrace{(\sum y_i - 1)}_{< 0} t \xrightarrow[t \to +\infty]{} +\infty$

  then $\underline{f^*(y) = +\infty}$

- now we suppose $y \succeq 0$ and $\sum_{i=1}^{m} y_i = 1$

  Let $x \in \mathbb{R}^n$. We note k such that $x_k = \max_i x_i$

  $$y^T x - f(x) = y_1 x_1 + \dots + (y_k - 1) x_k + \dots + y_n x_n$$
  $$= \underbrace{y_1}_{\geq 0} \underbrace{(x_1 - x_k)}_{\leq 0} + \dots + \underbrace{y_n}_{\geq 0} \underbrace{(x_n - x_k)}_{\leq 0}$$
  $$\leq 0$$

  and for $x = (1/n, \dots, 1/n)$ $\quad y^T x - f(x) = \frac{1}{n} \times 1 - \frac{1}{n} = 0$

  $\Rightarrow \underline{\sup (y^T x - f(x))} = 0$

Finally, $f^*(y) = \begin{cases} 0 & \text{if } y \succeq 0 \text{ and } \sum_{i=1}^{m} y_i = 1 \\ +\infty & \text{otherwise} \end{cases}$

(b) $f(x) = \sum_{i=1}^{r} x_{[i]} \quad x \in \mathbb{R}^n$

Let $y \in \mathbb{R}^n$

- if $\exists k \in \{1, \dots, n\}$ such that $y_k < 0$

  let $x_t = (0, \dots, 0, -t, 0, \dots, 0)$
  $\uparrow$ k^{th} component

  if $r = n$, $f(x) = -t \Rightarrow -f(x) \geq 0$ if $t \geq 0$

  else $f(x) = 0$

  therefore $y^T x - f(x) \geq y^T x = -y_k t \xrightarrow[t \to \infty]{} +\infty \Rightarrow f^*(y) = +\infty$

- by analogous reasoning than (a) we can prove that if $\sum_{i=1}^{n} y_i \neq r$ $f^*(y) = +\infty$ (using $x_t = (t, \dots, t)$ or $-(t, \dots, t)$)

- now we suppose $\sum_{i=1}^{n} y_i = r$ and $y \succeq 0$

$$y^T x - f(x) = (y_{[1]} - 1) x_{(1)} + \dots + (y_{[k]} - 1) x_{(k)} + \dots + y_{(n)} x_{(n)}$$

if $\exists k \in \{1, \dots, n\}$ such that $y_k > 1$, then

let $x_t = (0, 0, t, 0, \dots, 0)$
$\uparrow$ k^{th} component

$y^T x - f(x) = \underbrace{(y_k - 1)}_{> 0} t \xrightarrow[t \to +\infty]{} +\infty \Rightarrow f^*(y) = +\infty$

we suppose now that $y \preceq 1$

$y^T x - f(x) \leq (y_{[1]} - 1) x_{[k]} + \dots + [y_{[k]} - 1] x_{[k]} + \dots + y_{(n)} x_{(k)}$

$\leq x_{[k]} \left(\sum_{i=1}^{k} (y_{[i]} - 1) + \sum_{i=k+1}^{m} y_{[i]} \right)$

$\leq x_{[k]} \times 0 = 0$

and for $x = (1/n, \ldots, 1/n)$, $y^T x - f(x) = 0$

$\Rightarrow f^*(y) = 0$

Finally $f^*(y) = \begin{cases} 0 & \text{if } y \succeq 0,\ y \preceq 1 \text{ and } \sum_{i=1}^{m} y_i = r \\ +\infty & \text{otherwise} \end{cases}$

(c) $f(x) = \max_{i=1,\ldots,m} (a_i x + b_i)$ on $\mathbb{R}$, with $a_1 \leq \ldots \leq a_m$

When $x \to +\infty$, we have $\max_{i=1,\ldots,n} (a_i x + b_i) = a_m x + b_m$

hence if $y > a_m$, $f^*(y) = +\infty$

In the same way, if $y < a_1$ we can use $x \to -\infty$ to show $f^*(y) = +\infty$

Hence we now consider $y \in [a_1, a_m]$

We can note that f is piecewise linear (with the assumptions):

$$f(x) = \begin{cases} a_1 x + b_1 & \text{if } x \leq \frac{b_2 - b_1}{a_1 - a_2} \\ a_i x + b_i & \text{if } \frac{b_i - b_{i-1}}{a_{i-1} - a_i} \leq x \leq \frac{b_{i+1} - b_i}{a_i - a_{i+1}} \quad i = 2, \ldots, m-1 \\ a_m x + b_m & \text{otherwise} \end{cases}$$

Thus let $i = 1, \ldots, m$, and $y \in]a_i, a_{i+1}[$

let $g(x) = yx - f(x) = yx - \max (a_i x + b_i)$

- for $x < \frac{b_{i+1} - b_i}{a_i - a_{i+1}}$, g is strictly increasing since $y > a_i$ and f has a slope less than a_i

- for $x > \frac{b_{i+1} - b_i}{a_i - a_{i+1}}$, g is strictly decreasing since $y < a_{i+1}$ and f has a slope bigger than a_{i+1}

Therefore $g(x)$ reaches is maximum in $\frac{b_{i+1} - b_i}{a_i - a_{i+1}}$

and
$$g\left(\frac{b_{i+1} - b_i}{a_i - a_{i+1}}\right) = y\,\frac{b_{i+1} - b_i}{a_i - a_{i+1}} - \left(a_i \frac{b_{i+1} - b_i}{a_i - a_{i+1}} + b_i\right)$$
$$= (y - a_i)\,\frac{b_{i+1} - b_i}{a_i - a_{i+1}} - b_i$$

For $y = a_i$, the maximum is reached on all $x \in \left[\frac{b_i - b_{i-1}}{a_{i-1} - a_i}, \frac{b_{i+1} - b_i}{a_i - a_{i+1}}\right]$ so the previous formula holds

Therefore
$$f^*(y) = \begin{cases} (y - a_i)\,\frac{b_{i+1} - b_i}{a_i - a_{i+1}} - b_i & y \in [a_i, a_{i+1}] \\ +\infty & \text{otherwise} \end{cases}$$

(d) $f(x) = x^p$ on $\mathbb{R}_{++}$ where $p > 1$

Let $y \in \mathbb{R}$ and $g(x) = yx - x^p$

- g is differentiable and one has $g'(x) = y - p x^{p-1}$
- if $y < 0$, $g'(x) < 0$ so g is nonincreasing and reaches it sup for $x \to 0$. Since $g(x) \xrightarrow[x \to 0]{} 0$

$\Rightarrow f^*(y) = 0$

- if $y > 0$, $g'(x) = 0 \iff y = p\cdot x^{p-1} \iff x_0 = \left(\frac{y}{p}\right)^{1/p-1}$

and $\begin{cases} g'(x) > 0 & x \in]0, x_0[\\ g'(x) < 0 & x \in]x_0, +\infty[\end{cases}$

Therefore $\sup_{x \in \mathbb{R}_{++}} g(x) = g(x_0)$

$$g\left((y/p)^{1/p-1}\right) = y\,(y/p)^{1/p-1} - (y/p)^{p/p-1}$$
$$= (p-1)\,(y/p)^{p/p-1}$$

Finally $f^*(y) = \begin{cases} 0 & \text{if } y \leq 0 \\ (p-1)(y/p)^{p/p-1} & \text{if } y > 0 \end{cases}$

(d) where $p < 0$

- if $y \geq 0$, g is nondecreasing and for $x \to +\infty$, $g(x) \to +\infty$
  Therefore $f^*(y) = +\infty$
- if $y < 0$ we find the same result as before and
  $f^*(y) = (p-1)\,(y/p)^{p/p-1}$

Finally $f^*(y) = \begin{cases} (p-1)(y/p)^{p/p-1} & \text{if } y < 0 \\ +\infty & \text{if } y \geq 0 \end{cases}$

(e) $f(x) = -(\prod x_i)^{1/n}$ on $\mathbb{R}^n_{++}$

let $y \in \mathbb{R}$

- let $x_t = (1, \dots, 1, \underset{k^{th} \text{ component}}{t^n}, 1, \dots, 1)$ for $t \geq 0$

$$y^T x_t + (\prod x_i)^{1/n} = \sum_{i \neq k} y_i + t^n y_k + t$$

Therefore, if $\exists k \in \{1, \dots, n\}$ such that $y_k \geq 0$, then
$t^n y_k + t \underset{t \to +\infty}{\longrightarrow} +\infty$ and $f^*(y) = +\infty$

- now we suppose $y < 0$
  let $x_t = (-t/y_1, \dots, t/y_m)$. $x_t \in \mathbb{R}_{++}$ since $y < 0$

$$y^T x_t + (\Pi x_i)^{1/n} = -mt + t\left(\prod_{i=1}^{m} -\left(\frac{1}{y_i}\right)\right)^{1/n}$$

therefore, if $\left(\prod_{i=1}^{m} -1/y_i\right)^{1/n} > m$, $y^T x_T + (\Pi x_i)^{1/n} \xrightarrow[t \to +\infty]{} +\infty$

and $f^*(y) = +\infty$

· now we suppose that $\left(\prod_{i=1}^{m} -1/y_i\right)^{1/n} \leq m$

$g(x) = y^T x + (\Pi x_i)^{1/n} = \Sigma y_i x_i + (\Pi x_i)^{1/n}$

We have $\log\left(\Sigma \frac{1}{m}(-y_i)x_i\right) \geq \sum_{i=1}^{m} \frac{1}{m} \log(-y_i x_i)$ since log is concave

$$\geq \frac{1}{m} \log(\Pi -y_i x_i) = \frac{1}{m}\left[\log(\Pi -y_i) + \log(\Pi x_i)\right]$$

$$= \frac{1}{m}\left[\log(\Pi x_i) - \log\left(\Pi -\frac{1}{y_i}\right)\right]$$

$$= \log\left((\Pi x_i)^{1/n} / (\Pi -1/y_i)^{1/n}\right)$$

Therefore $\log\left(\Sigma \frac{1}{m}(-y_i x_i)\right) \geq \log\left((\Pi x_i)^{1/n} / \Pi(-1/y_i)^{1/n}\right)$

$$\Rightarrow -\frac{1}{m} \sum_{i=1}^{m} y_i x_i \geq (\Pi x_i)^{1/n} / \Pi(-1/y_i)^{1/n}$$

$$\Rightarrow -\frac{1}{m}\left(g(x) - (\Pi x_i)^{1/n}\right) \geq (\Pi x_i)^{1/n} / \Pi(-1/y_i)^{1/n}$$

$$\Rightarrow g(x) - (\Pi x_i)^{1/n} \leq -m (\Pi x_i)^{1/n} / \underbrace{\Pi(-1/y_i)^{1/n}}_{\leq m}$$

$$\leq -(\Pi x_i)^{1/n}$$

$$\Rightarrow g(x) \leq 0$$

and for $x_t = (t, \dots, t)$ ..., $g(x_t) \xrightarrow[t \to 0]{} 0 \Rightarrow g^*(y) = 0$

Finally $\quad f^*(y) = \begin{cases} 0 & \text{if } y < 0 \text{ and } \left(\prod_{i=1}^{m} -1/y_i\right)^{1/n} \leq n \\ +\infty & \text{otherwise} \end{cases}$

(f) $f(x,t) = -\log(t^2 - x^T x)$ for $(x,t) \in \{ \|x\|_2 < t \mid x \in \mathbb{R}^n, t \in \mathbb{R} \}$

Let $y = (y_x, y_t) \in \mathbb{R}^n \times \mathbb{R}$

- let $(x,t) = ((0,\dots,0), t)$, $t \in \mathbb{R}$

$y^T(x,t) - f(x,t) = y_t t + 2\log|t|$

therefore for $y_t \geq 0$ $\quad f^*(y) = +\infty$

- We suppose now that $y_t < 0$

let $g(x,t) = y_x^T x + y_t t + \log(t^2 - x^T x) \quad \sqrt{x^T x} < t$

One has:

$$\begin{cases} \frac{\partial g}{\partial x} = 0 \\ \frac{\partial g}{\partial t} = 0 \end{cases} \iff \begin{cases} y_x - 2x \frac{1}{t^2 - x^T x} = 0 \\ y_t + 2t \frac{1}{t^2 - x^T x} = 0 \end{cases} \iff \begin{cases} y_x = 2x \frac{1}{t^2 - x^T x} \\ \frac{1}{t^2 - x^T x} = -\frac{y_t}{2t} \end{cases}$$

$$\iff \begin{cases} y_x = -\frac{y_t}{t} x \\ 2t = -y_t(t^2 - x^T x) \end{cases} \iff \begin{cases} x = -\frac{t}{y_t} y_x \quad (\text{ok since } y_t \neq 0) \\ 2t = -y_t\left(t^2 + \frac{t^2}{y_t^2} y_x^T y_x\right) \end{cases}$$

$$\iff \begin{cases} y_t\left(t - \frac{t}{y_t^2} y_x^T y_x\right) + 2 = 0 \\ x = -\frac{t}{y_t} y_x \end{cases} \iff \begin{cases} t = -\frac{2 y_t}{y_t^2 - y_x^T y_x} \\ x = \frac{2 y_x}{y_t^2 - y_x^T y_x} \end{cases}$$

(and $\sqrt{x^T x} < t \iff \sqrt{y_x^T y_x} < -y_t$
$\iff \|y_x\| < -y_t$)

So there is a critical point if and only if $\|y_x\| < -y_t$

and the critical point is reached for $x = \dfrac{2y_x}{y_t^2 - y_x^T y_x}$

and $t = -\dfrac{2y_t}{y_t^2 - y_x^T y_x}$

for these values, $g(x,t) = \dfrac{2y_x^T y_x}{y_t^2 - y_x^T y_x} - \dfrac{2y_t^2}{y_t^2 - y_x^T y_x}$

$$+ \log(4) + \log\left(\frac{y_t^2}{(y_t^2 - y_x^T y_x)^2} - \frac{y_x^T y_x}{(y_t^2 - y_x^T y_x)^2}\right)$$

$$\Rightarrow g(x,t) = g_{0_y} = -2 + \log(4) - \underbrace{\log(|y_t^2 - y_x^T y_x|)}_{= \log(-y_t^2 + y_x^T y_x)}$$

- if $\|y_x\| < -y_t$
  - from Cauchy Schwarz inequality we have
    $y_x^T x \leq \|y_x\|\|x\| \leq t\|y_x\|$
    $\Rightarrow g(x,t) \leq t(\underbrace{\|y_x\| + y_t}_{<0 \text{ by assumption}}) + \log(t^2 - x^T x)$
  - When $\|x\| \to t$, $\log(t^2 - x^T x) \to -\infty$ so $g(x,t) \to -\infty$
  - When $\|(x,t)\| \to +\infty$, necessarily $t \to +\infty$, therefore since
    $g(x,t) \leq t(\|y_x\| + y_t) + \log(t^2)$, $g(x,t) \to -\infty$
  - $\{(x,t) \mid t \in \mathbb{R}, \|x\| < t\}$ is open
  - Hence $-g$ is coercive and continuous on an open set so g reaches its maximum. Since there is only one critical point, $f^*(y) = g_{0_y}$
- if $\|y_x\| \geq -y_t$, let $x_t = \dfrac{t-1}{\|y_x\|} y_x$ $(t \geq 1)$

$$g(x,t) = (t-1)\|y_x\| + y_t t + \log(t^2 - (t-1)^2)$$
$$\geq -\|y_x\| + \log(2t-1) \xrightarrow[t \to +\infty]{} +\infty \quad \text{so} \quad f^*(y) = +\infty$$

Finally: $f^*(y) = \begin{cases} -2 + \log 4 - \log(-y_t^2 + y_x^T y_x) & \text{if } y_t < 0, \|y_x\| < -y_t \\ +\infty & \text{otherwise} \end{cases}$