KERNEL METHODS IN ML

Homework 1

1.2 1.6 1.7 2 3

Exercise 1. Kernels

2. $\mathcal{X} = \mathbb{N}$ $\quad K(x, x') = 2^{x+x'}$

- Let $x, x' \in \mathbb{N}$, $\quad K(x, x') = 2^{x+x'} = 2^{x'+x} = K(x', x)$
- Let $N \in \mathbb{N}$, $(x_1, \dots, x_N) \in \mathbb{N}^N$ and $(a_1, \dots, a_N) \in \mathbb{R}^N$

$$\sum_{i=1}^{N} \sum_{j=1}^{N} a_i a_j K(x_i, x_j) = \sum_{i=1}^{N} \sum_{j=1}^{N} a_i a_j 2^{x_i} 2^{x_j} = \left(\sum_{i=1}^{N} a_i 2^{x_i} \right)^2 \geq 0$$

So K is positive definite

6. $\mathcal{X} = \mathbb{R}$ $\quad K(x, x') = \cos(x + x')$

- Take $N = 2$, $x_1 = x_2 = \pi/2$ and $a_1 = a_2 = 1$

One has $\sum_{i=1}^{N} \sum_{j=1}^{N} a_i a_j K(x_i, x_j) = 4 \cdot \cos(\pi) = -4 < 0$

So K is not positive definite

7. $\mathcal{X} = \mathbb{R}$ $\quad K(x, x') = \cos(x - x')$

- Let $x, x' \in \mathbb{R}$, $K(x, x') = \cos(x - x') = \cos(x' - x) = K(x', x)$
- Let $N \in \mathbb{N}$, $(x_1, \dots, x_N) \in \mathbb{R}^N$ and $(a_1, \dots, a_N) \in \mathbb{R}^N$

$$\sum_{i=1}^{N} \sum_{j=1}^{N} a_i a_j K(x_i, x_j) = \sum_{i=1}^{N} \sum_{j=1}^{N} a_i a_j \cos(x_i - x_j)$$

$$= \sum_{i=1}^{N} \sum_{j=1}^{N} a_i a_j (\cos x_i \cos x_j + \sin x_i \sin x_j)$$

$$= \left(\sum_{i=1}^{N} a_i \cos(x_i) \right)^2 + \left(\sum_{i=1}^{N} a_i \sin(x_i) \right)^2 \geq 0$$

So K is positive definite

Exercise 2. Function and kernel boundedness

Let $K : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$ a pd kernel s.t. $\forall x, z \in \mathcal{X}$, $K(x,z) \leq b^2$

Let $f \in \mathcal{H}$ such that $\|f\|_{\mathcal{H}} \leq 1$

Let $x \in \mathcal{X}$.

$$|f(x)| = |\langle f, K_x \rangle_{\mathcal{H}}| \leq \|f\|_{\mathcal{H}} . \|K_x\|_{\mathcal{H}} \quad \text{(Cauchy - Schwarz)}$$
$$\leq 1 . \sqrt{\langle K_x, K_x \rangle_{\mathcal{H}}} = \sqrt{K(x,x)}$$
$$\leq b$$

Therefore, $\sup_{x \in \mathcal{X}} |f(x)| \leq b \Rightarrow \underline{\|f\|_\infty \leq b}$

Exercise 3. Non-expansiveness of the Gaussian kernel

$$K : \mathbb{R}^p \times \mathbb{R}^p \to \mathbb{R} \qquad \varphi : \mathbb{R}^p \to \mathcal{H} \text{ s.t. } K(x,x') = \langle \varphi(x), \varphi(x') \rangle_{\mathcal{H}}$$
$$x, x' \mapsto e^{-\frac{\alpha}{2}\|x-x'\|^2}$$

Let $x \in \mathbb{R}^p$.

$$\|\varphi(x) - \varphi(x')\|_{\mathcal{H}} = \sqrt{\langle \varphi(x), \varphi(x) \rangle + \langle \varphi(x'), \varphi(x') \rangle - 2\langle \varphi(x), \varphi(x') \rangle}$$
$$= \sqrt{K(x,x) + K(x',x') - 2K(x,x')}$$
$$= \sqrt{2\left(1 - e^{-\frac{\alpha}{2}\|x-x'\|^2}\right)}$$

Denote $g : \mathbb{R}^+ \to \mathbb{R}$
$$t \mapsto \sqrt{2\left(1 - e^{-\frac{t^2}{2}}\right)}$$

One has $\forall t \in \mathbb{R}^+ ; 0 \geq e^{-\frac{t^2}{2}} - 1 \geq -\frac{t^2}{2} \Rightarrow 0 \leq 2\left(1 - e^{-t^2/2}\right) \leq t^2$
$$\Rightarrow g(t) \leq t$$

Therefore $\underline{\|\varphi(x) - \varphi(x')\|_{\mathcal{H}}} = g(\sqrt{\alpha}\|x-x'\|) \leq \underline{\sqrt{\alpha}\|x-x'\|}$