"Kernel methods in machine learning"
Homework 3
Due February 24, 2021, 3pm

Julien Mairal and Jean-Philippe Vert

Exercice 1. COCO

Given two vectors of real numbers $X = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $Y = (y_1, \ldots, y_n) \in \mathbb{R}^n$, the covariance between X and Y is defined as

$$cov_n(X, Y) = \mathbf{E}_n(XY) - \mathbf{E}_n(X)\mathbf{E}_n(Y)\,,$$

where $\mathbf{E}_n(U) = (\sum_{i=1}^n u_i)/n$. The covariance is useful to detect linear relationships between X and Y. In order to extend this measure to potential nonlinear relationships between X and Y, we consider the following criterion:

$$C_n^K(X, Y) = \max_{f,g \in \mathcal{B}_K} cov_n(f(X), g(Y))\,,$$

where K is a positive definite kernel on $\mathbb{R}$, $\mathcal{B}_K$ is the unit ball of the RKHS of K, and $f(U) = (f(u_1), \ldots, f(u_n))$ for a vector $U = (u_1, \ldots, u_n)$.

1. Express simply $C_n^K(X, Y)$ for the linear kernel $K(a, b) = ab$.

2. For a general kernel K, express $C_n^K(X, Y)$ in terms of the Gram matrices of X and Y.